ON STARLIKENESS AND CONVEXITY OF CERTAIN MULTIVALENT FUNCTIONS(Topics in Univalent Functions and Its Applications)

YAGUCHI, TERUO

数理解析研究所講究録
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TERUO YAGUCHI (日大文理 谷口彰男)

Abstract

The object of the present paper is to determine the radii of starlikeness and convexity of order α of certain analytic multivalent functions with a kind of bounded argument.

1. Introduction

Let p, α, β and r denote $p \in N = \{1,2,3,\cdots\}$, $0 \leq \alpha < p$, $\beta > 0$ and $0 < r \leq 1$, respectively. Let U_r denote the set $\{z: |z| < r\}$ and let U denote the unit disk U_1. Next, let A_p denote the class of functions of the form :

$$(1.1) \qquad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

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which are analytic in the unit disk U. We denote A_1 by A.

A function $f(z)$ in the class A_p is said to be p-valently starlike of order α in U_r if and only if it satisfies

$$\text{(1.2)} \qquad \operatorname{Re}\left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \qquad (z \in U_r).$$

We denote by $S_p^*(\alpha)_r$ the subclass of the class A_p consisting of all p-valently starlike functions of order α in U_r.

Further, a function $f(z)$ in the class A_p is said to be p-valently convex of order α in U_r if and only if it satisfies

$$\text{(1.3)} \qquad \operatorname{Re}\left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \qquad (z \in U_r).$$

Also, we denote by $K_p(\alpha)_r$ the subclass of the class A_p consisting of all p-valently convex functions of order α in U_r.

Let M be the class of functions of the form :

$$\text{(1.4)} \qquad p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

which are analytic in the unit disk U.

A function $p(z)$ in the class M is said to be a member of the class $M(\beta)$ if and only if it satisfies

$$\text{(1.5)} \qquad | \arg p(z) | < \frac{\pi \beta}{2} \qquad (z \in U).$$

Finally, a function $f(z)$ in the class A_p is said to be p-valent strongly close-to-convex of order α and type β in U_r if and only if there is a function $g(z) \in K_p(\alpha)_r$ such that $\frac{f'(z)}{g'(z)} \in M(\beta)$. We denote by $C_p(\alpha, M_\beta)_r$ the subclass of the class A_p consiting of all p-valent strongly close-to-convex functions of order α and type β in U_r.

In particular, whenever the numbers p, α, β and r mentioned in technical terms $S_p^*(\alpha)_r$, $K_p(\alpha)_r$ and $C_p(\alpha, M_\beta)_r$ are equal to 1, 0, 1 and 1, respectively, these numbers are removed from the technical terms. For example,

$$S_p^*(\alpha) = S_p^*(\alpha)_1, \quad K_p(\alpha) = K_p(\alpha)_1, \quad S^*(\alpha)_r = S_1^*(\alpha)_r,$$

$$S^*(\alpha) = S_1^*(\alpha)_1, \quad K(\alpha)_r = K_1(\alpha)_r, \quad C_p(\alpha, M_\beta) = C_p(\alpha, M_\beta)_1,$$

$$S^* = S_1^*(0)_1, \quad K = K_1(0)_1, \quad C = C_1(0, M_1)_1.$$

A function $f(z)$ in the classes S^*, K and C is said to be starlike, convex and close-to-convex, respectively.

2. The radii of starlikeness

In order to get our results, we here have to recall Lemma 2.A and prove Lemma 2.1.

Lemma 2.A (Nunokawa and Causey [1]). *Let β be $\beta > 0$. If $p(z) \in M(\beta)$, then*

$$(2.1) \qquad \left| \frac{p'(z)}{p(z)} \right| \leq \frac{2\beta}{1 - |z|^2} \qquad (z \in U).$$

Lemma 2.1. *Let p and α be $p \in N$ and $0 \leq \alpha < p$, respectively.*

If $g(z) \in S_p^*(\alpha)$, *then*

$$\text{(2.2)} \qquad \mathrm{Re}\left\{ \frac{z\, g'(z)}{g(z)} \right\} \geq (p-\alpha)\frac{1-|z|}{1+|z|} + \alpha \qquad (z \in U).$$

Proof. Defining the functions $h(z)$ and $h_0(z)$ by

$$\text{(2.3)} \qquad h(z) = \frac{z\, g'(z)}{p\, g(z)} \quad \text{and} \quad h_0(z) = \left(1 - \frac{\alpha}{p}\right)\frac{1-z}{1+z} + \frac{\alpha}{p},$$

respectively, we have

$$h(z),\ h_0(z) \in M, \quad h(0) = h_0(0) \quad \text{and} \quad \mathrm{Re}\, h(z) > \frac{\alpha}{p},$$

because of $g(z) \in S_p^*(\alpha)$. Since the function $h_0(z)$ is univalent in U and maps the unit disk U onto $\mathrm{Re}\, w > \frac{\alpha}{p}$, we obtain

$$\left| h_0^{-1}(h(z)) \right| \leq |z| \qquad (z \in U)$$

by using Schwarz's lemma. This inequality shows that the image of the unit disk U by $h(z)$ have to be in the disk whose diameter end points are

$$\left(1 - \frac{\alpha}{p}\right)\frac{1-|z|}{1+|z|} + \frac{\alpha}{p} \quad \text{and} \quad \left(1 - \frac{\alpha}{p}\right)\frac{1+|z|}{1-|z|} + \frac{\alpha}{p}.$$

This completes the proof of Lemma 2.1. *q.e.d.*

Now, we have

Theorem 2.1. *Let* p, j, α, β *and* γ *be* $p \in N$, $j = 0, 1, 2, \cdots, p-1$, $0 \leq \alpha < p-j$, $\beta > 0$ *and* $0 \leq \gamma < p-j$, *respectively. If a function* $f(z)$ *is in the class* A_p *and*

$$\frac{f^{(j)}(z)}{g^{(j)}(z)} \in M(\beta)$$

for some $g(z) \in A_p$ *such that*

$$\frac{(p-j)!}{p!} g^{(j)}(z) \in S^*_{p-j}(\alpha),$$

then

$$\frac{(p-j)!}{p!} f^{(j)}(z) \in S^*_{p-j}(\gamma)_r,$$

where

$$(2.4)\qquad \begin{cases} r = \dfrac{p-j-\alpha+\beta-\sqrt{A}}{p-j-2\alpha+\gamma} & \text{if } p-j-2\alpha+\gamma > 0, \\[2ex] r = \dfrac{p-j-\alpha+\beta+\sqrt{A}}{p-j-2\alpha+\gamma} & \text{if } p-j-2\alpha+\gamma < 0, \\[2ex] r = \dfrac{p-j-\alpha}{p-j-\alpha+\beta} & \text{if } p-j-2\alpha+\gamma = 0, \end{cases}$$

and

$$A = (\alpha-\beta-\gamma)^2 + 2\beta\,(p-j-\gamma).$$

The result is sharp for the function $f(z)$ *defined by*

$$(2.5)\qquad f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}}\left(\frac{1+z}{1-z}\right)^{\beta} \qquad \text{as} \quad j = 0,$$

$$(2.6)\qquad f(z) = \int_0^z \int_0^{\xi_j} \cdots \int_0^{\xi_2} \frac{p!}{(p-j)!} \frac{\xi_1^{p-j}}{(1-\xi_1)^{2(p-j-\alpha)}} \left(\frac{1+\xi_1}{1-\xi_1}\right)^{\beta} d\xi_1 \cdots d\xi_{j-1} d\xi_j$$

$$\text{as} \quad j = 1, 2, \cdots, p-1,$$

at $z = -|z|$.

Proof. Defining the function $p(z)$ by

$$(2.7)\qquad p(z) = \frac{f^{(j)}(z)}{g^{(j)}(z)},$$

we have $p(z) \in M(\beta)$. Since

$$(2.8) \qquad \frac{p'(z)}{p(z)} = \frac{f^{(j+1)}(z)}{f^{(j)}(z)} - \frac{g^{(j+1)}(z)}{g^{(j)}(z)},$$

we see, using Lemma 2.A, that

$$(2.9) \qquad \left| \frac{z\, f^{(j+1)}(z)}{f^{(j)}(z)} - \frac{z\, g^{(j+1)}(z)}{g^{(j)}(z)} \right| \leq \frac{2\beta\, |z|}{1 - |z|^2} \qquad (z \in U).$$

The function $\frac{(p-j)!}{p!} g^{(j)}(z) \in S^*_{p-j}(\alpha)$ satisfies

$$(2.10) \qquad \mathrm{Re}\left\{ \frac{z\, g^{(j+1)}(z)}{g^{(j)}(z)} \right\} \geq (p - j - \alpha) \frac{1 - |z|}{1 + |z|} + \alpha \qquad (z \in U),$$

by Lemma 2.1. Therefore, it follows from (2.9) and (2.10) that

$$(2.11) \qquad \mathrm{Re}\left\{ \frac{z\, f^{(j+1)}(z)}{f^{(j)}(z)} \right\} \geq \mathrm{Re}\left\{ \frac{z\, g^{(j+1)}(z)}{g^{(j)}(z)} \right\} - \frac{2\beta\, |z|}{1 - |z|^2}$$

$$\geq (p - j - \alpha) \frac{1 - |z|}{1 + |z|} + \alpha - \frac{2\beta\, |z|}{1 - |z|^2}$$

$$> \gamma, \text{ in } U_r.$$

q.e.d.

We here obtain two corollaries. Putting $j = 0$ in Theorem 2.1, we have Corollary 2.1.

Corollary 2.1. *Let* p, α, β *and* γ *be* $p \in N$, $0 \leq \alpha < p$, $\beta > 0$ *and* $0 \leq \gamma < p$, *respectively. If a function* $f(z)$ *is in the class* A_p *and*

$$\frac{f(z)}{g(z)} \in M(\beta)$$

for some $g(z) \in S^*_p(\alpha)$, *then*

$$f(z) \in S_p^*(\gamma)_r,$$

where

$$(2.12)\quad \begin{cases} r = \dfrac{p - \alpha + \beta - \sqrt{B}}{p - 2\alpha + \gamma} & \text{if} \quad p - 2\alpha + \gamma > 0, \\[2ex] r = \dfrac{p - \alpha + \beta + \sqrt{B}}{p - 2\alpha + \gamma} & \text{if} \quad p - 2\alpha + \gamma < 0, \\[2ex] r = \dfrac{p - \alpha}{p - \alpha + \beta} & \text{if} \quad p - 2\alpha + \gamma = 0, \end{cases}$$

and

$$B = (\alpha - \beta - \gamma)^2 + 2\beta\,(p - \gamma).$$

The result is sharp for the function $f(z)$ *defined by* (2.6) *at* $z = -|z|$.

Putting $p = 1$ in Corollary 2.1, we have Corollary 2.2.

Corollary 2.2. *Let* α, β *and* γ *be* $0 \leq \alpha < 1$, $\beta > 0$ *and* $0 \leq \gamma < 1$, *respectively. If a function* $f(z)$ *is in the class* A *and*

$$\frac{f(z)}{g(z)} \in M(\beta)$$

for some $g(z) \in S^*(\alpha)$, *then*

$$f(z) \in S^*(\gamma)_r,$$

where

$$(2.13)\quad \begin{cases} r = \dfrac{1 - \alpha + \beta - \sqrt{C}}{1 - 2\alpha + \gamma} & \text{if} \quad 1 - 2\alpha + \gamma > 0, \\[2ex] r = \dfrac{1 - \alpha + \beta + \sqrt{C}}{1 - 2\alpha + \gamma} & \text{if} \quad 1 - 2\alpha + \gamma < 0, \\[2ex] r = \dfrac{1 - \alpha}{1 - \alpha + \beta} & \text{if} \quad 1 - 2\alpha + \gamma = 0, \end{cases}$$

and

$$C = (\alpha - \beta - \gamma)^2 + 2\beta\ (1 - \gamma).$$

The result is sharp for the function $f(z)$ *defined by*

$$(2.14) \qquad f(z) = \frac{z}{(1 - z)^{2(1-\alpha)}} \left(\frac{1 + z}{1 - z} \right)^{\beta}$$

at $z = -|z|$.

Remark 2.1. Taking $\alpha = 0$ in Corollary 2.2, we have the corresponding result due to Yaguchi, Obradović, Nunokawa and Owa [3]. Furthermore, taking $\gamma = 0$ and $\beta = 2$ in Corollary 2.2, we have the corresponding result due to Yaguchi and Nunokawa [2].

3. The radius of convexity

Noting that $f(z) \in K_p(\alpha)$ if and only if $zf'(z) \in S_p^*(\alpha)$, we get the following result with the aid of Theorem 2.1.

Theorem 3.1. *Let* p, j, α, β *and* γ *be in the same conditions as in Theorem 2.1. If a function* $f(z)$ *is in the class* A_p *and*

$$\frac{f^{(j+1)}(z)}{g^{(j+1)}(z)} \in M(\beta)$$

for some $g(z) \in A_p$ *such that*

$$\frac{(p - j)!}{p!} g^{(j)}(z) \in K_{p-j}(\alpha),$$

then

$$\frac{(p - j)!}{p!} f^{(j)}(z) \in K_{p-j}(\gamma)_r,$$

where r *is given by* (2.4). *The result is sharp for the function* $f(z)$ *defined by*

$$(3.1)\quad f(z)=\int_0^z\int_0^{\xi_j}\cdots\int_0^{\xi_1}\frac{p!\qquad \xi_0^{p-j-1}}{(p-j-1)!(1-\xi_0)^{2(p-j-\alpha)}}\left(\frac{1+\xi_0}{1-\xi_0}\right)^{\beta}d\xi_0\cdots d\xi_{j-1}d\xi_j,$$

at $z=-|z|$.

Proof. Defining the functions $F(z)$ and $G(z)$ by

$$F(z)=\frac{(p-j-1)!}{p!}z\,f^{(j+1)}(z)$$

and

$$G(z)=\frac{(p-j-1)!}{p!}z\,g^{(j+1)}(z),$$

respectively. We have

$$F(z)\in A_{p-j},\quad G(z)\in S^*_{p-j}(\alpha)\quad\text{and}\quad\frac{F(z)}{G(z)}\in M(\beta).$$

By Corollary 2.1, we obtain that $F(z)\in S^*_{p-j}(\gamma)_r$, where r is given by (2.2). Therefore $\frac{(p-j)!}{p!}f^{(j)}(z)$ is $(p-j)$-valently convex of order γ in U_r. *q.e.d.*

Putting $j=0$ in Theorem 3.1, we have Corollary 3.1.

Corollary 3.1. *Let* p, α, β *and* γ *be in the same conditions as in Corollary 2.1. If a function* $f(z)$ *in* A_p *is in the class* $C_p(\alpha,M_\beta)$, *then* $f(z)\in K_p(\gamma)_r$, *where* r *is given by (2.12). The result is sharp for the function* $f(z)$ *defined by*

$$(3.2)\qquad f(z)=\int_0^z\frac{p\,\xi^{p-1}}{(1-\xi)^{2(p-\alpha)}}\left(\frac{1+\xi}{1-\xi}\right)^{\beta}d\xi,$$

at $z=-|z|$.

Putting $p=1$ in Corollary 3.1, we have Corollary 3.2.

Corollary 3.2. *Let α, β and γ be in the same conditions as in Corollary 2.2. If a function $f(z)$ in the class A is in the class $C(\alpha, M_\beta)$, then $f(z) \in K(\gamma)_r$, where r is given by (2.13). The result is sharp for the function $f(z)$ defined by*

$$(3.3) \qquad f(z) = \int_0^z \frac{1}{(1-\xi)^{2(1-\alpha)}} \left(\frac{1+\xi}{1-\xi} \right)^\beta d\xi$$

at $z = -|z|$.

Putting $\beta = 1$ in Corollary 3.2, we have Corollary 3.3.

Corollary 3.3. *If a function $f(z)$ in the class A is close-to-convex of order α $(0 \leq \alpha < 1)$, then $f(z)$ is convex of order γ in U_r, where*

$$(3.4) \qquad \begin{cases} r = \dfrac{2 - \alpha - \sqrt{D}}{1 - 2\alpha + \gamma} & \text{if} \quad 1 - 2\alpha + \gamma > 0, \\[2ex] r = \dfrac{2 - \alpha + \sqrt{D}}{1 - 2\alpha + \gamma} & \text{if} \quad 1 - 2\alpha + \gamma < 0, \\[2ex] r = \dfrac{1 - \alpha}{2 - \alpha} & \text{if} \quad 1 - 2\alpha + \gamma = 0, \end{cases}$$

and

$$D = (\alpha - \gamma)^2 - 2\alpha + 3.$$

The result is sharp for the function $f(z)$ defined by

$$(3.5) \qquad f(z) = \frac{1}{(2\alpha - 1)(1 - \alpha)} \left(\frac{\alpha - (1 - \alpha)z}{(1 - z)^{2(1-\alpha)}} - \alpha \right) \qquad \left(\alpha \neq \frac{1}{2} \right)$$

$$(3.6) \qquad f(z) = \frac{2z}{1 - z} + \log(1 - z) \qquad \left(\alpha = \frac{1}{2} \right)$$

at $z = -|z|$.

Putting $\alpha = 0$ in Corollary 3.3, we have Corollary 3.4.

Corollary 3.4. *If a function $f(z)$ in the class A is close-to-convex, then $f(z)$ is convex of order γ in U_r,*

where $r = \dfrac{2 - \sqrt{\gamma^2 + 3}}{1 + \gamma}$.

The result is sharp for the Koebe function

$$f(z) = \frac{z}{(1 - z)^2}$$

at $z = -|z|$.

References

[1] Nunokawa,M. and Causey,W.M.: On certain analytic functions bounded argument, Sci. Rep. Fac. Edu. Gunma Univ., 34 (1985), 1 - 3.

[2] Yaguchi,T. and Nunokawa,M.: Functions whose derivatives do not assume non-positive real values, Proc. Inst. Nat. Sci. Nihon Univ., 23 (1988), 83 - 85.

[3] Yaguchi,T.,Obradovic,M.,Nunokawa,M. and Owa,S.: On starlikeness and convexity of certain analytic functions, Proc. Inst. Nat. Sci. Nihon Univ., 24 (1989), 117 - 120.

Department of Mathematics
College of Humanities and Sciences
Nihon University
3-25-40 Sakurajousui, Setagaya,
Tokyo, 156 JAPAN